Application of generalized Pythagoras theorem to calculation of configuration factors between surfaces of channels of revolution

Czesław Buraczewski and Jan Stąsiek*

The generalized Pythagoras theorem is used to derive new forms of formulae for configuration factors in cylindrical and conical channels. The equations are then approximated by an exponential function which simplifies the calculation of radiation heat transfer in nonisothermal enclosures.

Keywords: *Radiative heat transfer, Pythagoras theorem, configuration factors*

Because the determination of configuration factors is often very difficult, an attempt has been made to determine the factors for spatial systems, such as cylindrical and conical channels, using the generalized Pythagoras theorem. The equations derived for configuration factors are in some cases much simpler than those found in the literature[1–4], but the results are the same. The configuration factors $dF_{d1,d2}$ between the elementary surfaces $dF_{1,d2}$ and $F_{d2,1}$ are derived by differentiating the equation for $F_{1,2}$, the configuration factor for the specified surfaces.

The resulting equations are approximated by the exponential function $G(X)$ which, in comparison with the function proposed by Buckley[5], is more general and includes both cylindrical and conical channels. However, the main purpose of the paper is the presentation of the new theorem and its application to the calculation of configuration factors.

Generalized pythagoras theorem

Consider a triangle with sides B, C and M and angles α, β and γ (Fig 1). The relation between the sides and the cosines of the angles can be expressed by Carnot's theorem. For example, for side M we can write:

$$M^2 = C^2 + B^2 - 2CB \cos \gamma \qquad (1)$$

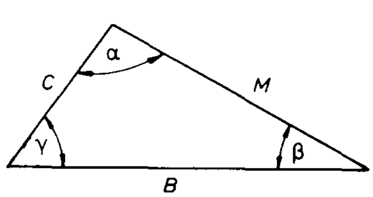

Fig 1

Fig 2(a) shows an isosceles trapezium† (1–4–2–3) on the base B. The intersection of the trapezium height h on the base produces a segment of length k:

$$k = (B - b)/2 \qquad (2)$$

Cos γ is given by:

$$\cos \gamma = k/C = (B - b)/2C \qquad (3)$$

From Eqs (1) and (3):

$$M^2 = C^2 + B^2 - 2CB\left(\frac{B-b}{2C}\right) = C^2 + Bb \qquad (4)$$

Similarly, for the other sides (Fig 2(b)) one may write:

$$C^2 = B^2 + Mm \qquad (5)$$

$$B^2 = M^2 + Cc \qquad (6)$$

These relations were obtained on the assumption that $B > 0$, $C > 0$, $M > 0$, $b \geq 0$, $c \geq 0$ and $m \geq 0$[1].

This new relation between the sides of a triangle and the bases of trapeziums built on those sides has been called the generalized Pythagoras theorem.

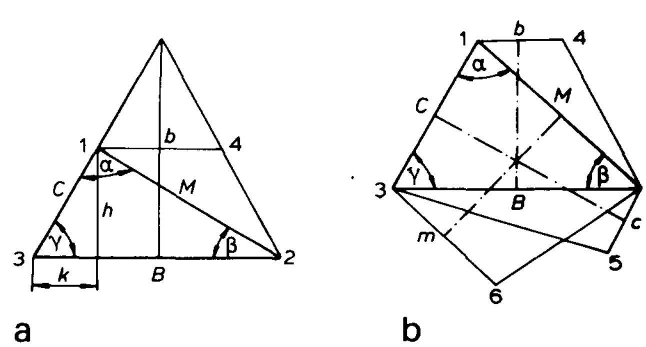

Fig 2

* Technical University of Gdansk, Institute of Heat Technics, 80-952 Gdansk, Poland
Received 24 January 1983 and accepted for publication on 25 May 1983
† In this paper 'trapezium' is used in the English sense to mean a quadrilateral with one pair of sides parallel

Determination of the configuration factor $F_{1,2}$ between the circular surfaces whose centres lie on a common normal line

From radiation heat transfer theory, for a system of surfaces A_1 and A_2 (Fig 3) we have the following[1]:

$$A_1F_{1,2}=A_2F_{2,1}$$
$$=\frac{\pi}{2}\{R^2+r^2+h^2-[(R^2+r^2+h^2)^2$$
$$-4R^2r^2]^{1/2}\} \quad (7)$$

Fig 3(b) shows an axial section of the enclosure, from which we can see that:

$$x^2=(R-r)^2+h^2 \quad (8)$$

or

$$R^2+r^2+h^2=x^2+2Rr \quad (9)$$

From Eqs (7) and (9):

$$A_1F_{1,2}=A_2F_{2,1}$$
$$=\frac{\pi}{2}\{x^2+2Rr$$
$$-[(x^2+2Rr)^2-4R^2r^2]^{1/2}\} \quad (10)$$

or, after transformation,

$$A_1F_{1,2}=A_2F_{2,1}=\pi\left\{\frac{[x^2+Dd]^{1/2}-x}{2}\right\}^2 \quad (11)$$

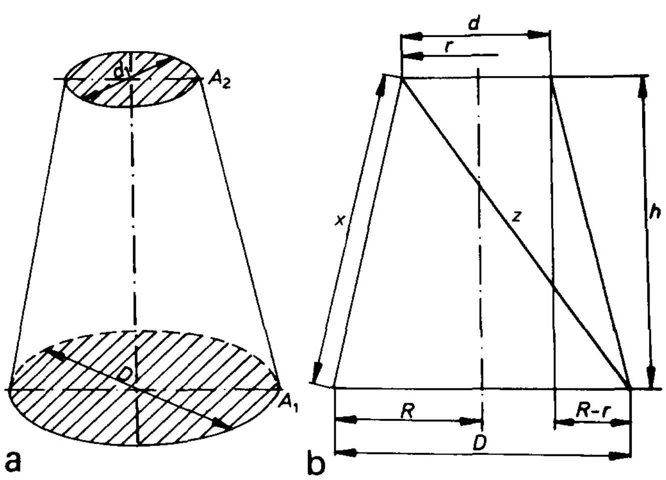

Fig 3 Cone frustum enclosure

From the generalized Pythagoras theorem of Eq (4) it can be seen that the root in Eq (11) is the diagonal z (of the enclosure shown in Fig 3); that is:

$$z^2=x^2+Dd \quad (12)$$

or

$$z=(x^2+Dd)^{1/2} \quad (13)$$

and Eq (11) takes the following form:

$$A_1F_{1,2}=A_2F_{2,1}=\frac{\pi(z-x)^2}{4} \quad (14)$$

Then, as:

$$A_1=\pi D^2/4 \quad (15)$$

$$A_2=\pi d^2/4 \quad (16)$$

the final equations are obtained:

$$F_{1,2}=\left(\frac{z-x}{D}\right)^2 \quad (17)$$

$$F_{2,1}=\left(\frac{z-x}{d}\right)^2 \quad (18)$$

It can therefore be seen that knowledge of the characteristic geometrical dimensions of the enclosure is quite sufficient for calculating the configuration factors in Eqs (17) and (18).

Configuration factors $F_{1,2}$, $dF_{1,d2}$, $F_{d2,1}$ and $dF_{d1,d2}$ which apply to cylindrical channels

According to Eq (17) and with the nomenclature shown on Fig 4, the configuration factor $F_{1,2}$ between two parallel and equal surfaces A_1 and A_2 is:

$$F_{1,2}=F_{2,1}=\frac{1}{D^2}\{[(x_2-x_1)^2+D^2]^{1/2}-(x_2-x_1)\} \quad (19)$$

where

$$[(x_2-x_1)^2+D^2]^{1/2}=Z \quad (20)$$

When we put:

$$x_1=0, (x_2-x_1)=x \quad \text{and } X=\frac{x}{D} \quad (21$$

Eq (19) takes the form:

$$F_{1,2}=F_{2,1}=([X^2+1]^{1/2}-X)^2 \quad (22)$$

Notation

A, dA	Area, elementary area
D, d	Channel diameter
R, r	Radius
F, dF	Configuration factors
$G(X)$	Exponential function
$K(X)$	Kernel in Eq (41)
x	Coordinate
Z	Diagonal of cylindrical or conical channel
$X=x/D$	Dimensionless diameter
$a=D/L$	Dimensionless parameter characteristic for conical channel
L	Ruling of cone
B, C, M / b, c, m, k, h	Characteristic dimensions of triangle or isosceles trapezium
1, 2, 3 / 4, 5, 6	Characteristic points of triangle or isosceles trapezium
α, β, γ	Triangle angles or constants in function $G(X)$
1, 2	Indexes of characteristic areas, coefficients or lengths

For simple geometrical shapes like cylindrical or conical channels, the configuration factors between A_1 and dA_2: dA_2 and A_1; and dA_1 and dA_2 are given by the following equations[4]

$$dF_{1,d2} = -\frac{\partial F_{1,2}}{\partial x_2}\,dx_2 \tag{23}$$

$$F_{d2,1} = -\frac{A_1}{dA_2}\frac{\partial F_{1,2}}{\partial x_2}\,dx_2 \tag{24}$$

$$dF_{d2,d1} = -\frac{A_1}{dA_2}\frac{\partial^2 F_{1,2}}{\partial x_1\,\partial x_2}\,dx_1\,dx_2 \tag{25}$$

or

$$dF_{d1,d2} = -\frac{A_1}{dA_1}\frac{\partial^2 F_{1,2}}{\partial x_1\,\partial x_2}\,dx_1\,dx_2 \tag{26}$$

Following differentiation of Eq (19) with respect to x_2, (23) takes the following form:

$$dF_{1,d2} = \frac{2[(x^2+D^2)^{1/2}-x]^2}{D^2(x^2+D^2)^{1/2}}\,dx_2 \tag{27}$$

On introducing the non-dimensional quantities $X = x/D$ and $dX_2 = dx_2/D$, then we have:

$$dF_{1,d2} = \left[\frac{4X^2+2}{(X^2+1)^{1/2}} - 4X\right] dX_2 \tag{28}$$

or

$$dF_{1,d2} = \frac{2}{(X^2+1)^{1/2}}\,F_{1,2}\,dx_2 \tag{29}$$

We adopt a similar procedure to calculate $F_{d2,1}$ from Eq (24):

$$F_{d2,1} = \frac{X^2+\frac{1}{2}}{(X^2+1)^{1/2}} - X \tag{30}$$

or

$$F_{d2,1} = \frac{1}{2(X^2+1)^{1/2}}\,F_{1,2} \tag{31}$$

Calculation of configuration factor $dF_{d1,d2}$ requires double differentiation of Eq (19), once with respect to x_1 and then with respect to x_2; hence:

$$\frac{\partial F_{1,2}}{\partial x_1} = \frac{2\{[(x_2-x_1)^2+D^2]^{1/2}-(x_2-x_1)\}\times\{[(x_2-x_1)^2+D^2]^{1/2}-(x_2-x_1)\}}{D^2[(x_2-x_1)^2+D^2]^{1/2}} \tag{32}$$

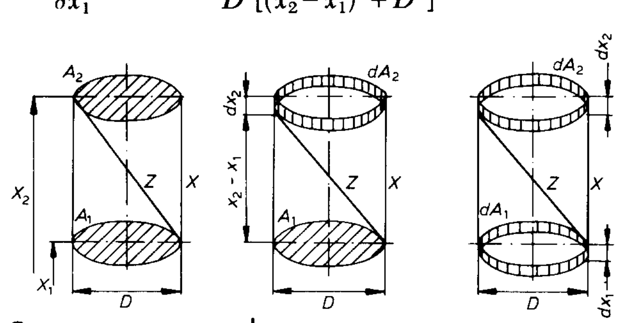

Fig 4 Cylindrical channel: (a) two real surfaces, (b) a real and elementary surface, (c) two elementary surfaces

Assume $x_1 = 0$ and differentiate eq (32) with respect to x_2 to obtain:

$$\frac{\partial^2 F_{1,2}}{\partial x_1\,\partial x_2} = \frac{4}{D^2}\left\{\frac{2x^3+3xD^2}{2[x^2+D^2]^{3/2}} - 1\right\} dx_1\,dx_2 \tag{33}$$

where $x = (x_2 - x_1)$. Eq (33) is then substituted into Eq (25) to yield the final result:

$$dF_{d1,d2} = \left[1 - \frac{2X^3+3X}{2(X^2+1)^{3/2}}\right] dX_2 \tag{34}$$

where $X = x/D$ and $dX_2 = dx_2/D$.

Configuration factors $F_{1,2}$, $dF_{1,d2}$, $F_{d2,1}$ and $dF_{d1,d2}$ which apply to conical channels

Configuration factors for conical channels (Fig 5) may be calculated by the same procedure as for cylindrical channels, using Eqs (13) and (17), and letting one of the diameters vary as a function of the variable x; ie (Fig 5(a)):

$$d = \left(1 - \frac{x}{L}\right) D \tag{35}$$

Then, from Eqs (35) and (12):

$$Z^2 = x^2 + D^2\left(\frac{L-x}{L}\right) \tag{36}$$

If we substitute Eq (36) into Eq (17), the result is:

$$F_{1,2} = \frac{1}{D^2}\left\{\left[(x_2-x_1)^2 + D^2\left(1 - \frac{x_2-x_1}{L}\right)\right]^{1/2} - (x_2-x_1)\right\}^2 \tag{37}$$

For $x_1 = 0$; $(x_2 - x_1) = X$ and $X = x/D$, and with $a = D/L$, Eq (37) takes the following form:

$$F_{1,2} = \{[X^2 + (1 - aX)]^{1/2} - X\}^2 \tag{38}$$

Eq (38) is more general than Eq (22) as the characteristic dimensionless coefficient a has been introduced.

For $a = 0$ $(L \to \infty)$, Eq (38) becomes similar to Eq (22).

Following the same calculation procedure as for cylindrical channels, and using Eqs (23), (24), (25)

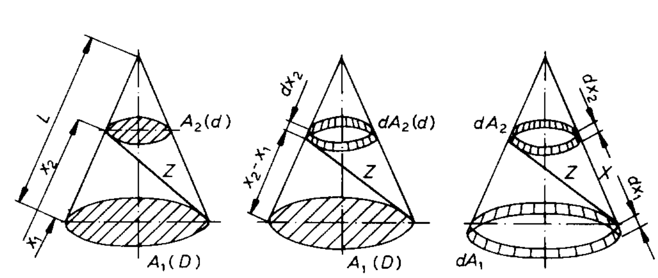

Fig 5 Conical channel: (a) two real surfaces, (b) a real and elementary surface, (c) two elementary surfaces

and (37), we can obtain the following configuration factors:

$$dF_{1,d2}=\frac{\dfrac{\{[X^2+(1-aX)]^{1/2}-X\}}{\{[X^2+(1-aX)]^{1/2}-X+0.5a\}}}{[X^2+(1-aX)]^{1/2}}\,dX_2 \tag{39}$$

$$F_{d2,1}=\frac{\dfrac{\{[X^2+(1-aX)]^{1/2}-X\}}{\{[X^2+(1-aX)]^{1/2}-X+0.5a\}}}{2(1-aX)[X^2+(1-aX)]^{1/2}} \tag{40}$$

and

$$dF_{d1,d2}=\left[1-\frac{2X^3-3aX^2+X(3+0.75a^2)-a}{2[X^2+(1-aX)]^{3/2}}\right]dX_2 \tag{41}$$

or

$$dF_{d1,d2}=K(X)\,dX_2 \tag{42}$$

where $K(X)$ is the kernel in Eq (41), and $X=x/D$, $a=D/L$ and $dX_2=dx_2/D$.

Approximation of configuration factors $F_{d2,1}$ and $dF_{d1,d2}$ with the exponential function $G(X)$

Because of the complicated forms of Eqs (30), (40), (34) and (41), their direct application to the calculation of radiation heat transfer may be difficult or sometimes impossible. However, according to Buckley[5] and Usiskin and Siegel[6] these equations may be succesfully approximated with an exponential function of the following type:

$$G(X)=\alpha\exp(-\beta X^{\gamma})$$

Fig 6 shows the function $F_{d2,1}$ and the kernel $K(X)$ in Eq (41) for several values of the parameter a. In this case the kernel was approximated by the function:

$$G(X)=\left(1+\frac{a}{2}\right)\exp(-2[X^{(1+a/2)}]) \tag{43}$$

It has also been found that the function $G(X)$ can satisfactorily approximate Eq (40) according to the formula:

$$2F_{d2,1}=G(X) \tag{44}$$

An analysis of the plots shown on Fig 6 leads to the conclusion that for parameters $a \gg 0$ the differences between the exact and approximate values are considerable and take the highest value when $x=L$.

Conclusions

The generalized Pythagoras theorem has been used to derive new forms of equations for configuration factors, and these are given in Eqs (17), (18), (22), (29) and (31) and Eqs (38), (39), (40) and (41). These are simpler than those found previously, and the calculation process is shortened.

Application of the generalized Pythagoras theorem to other geometrical shapes has also proved to give good results, as shown elsewhere[7].

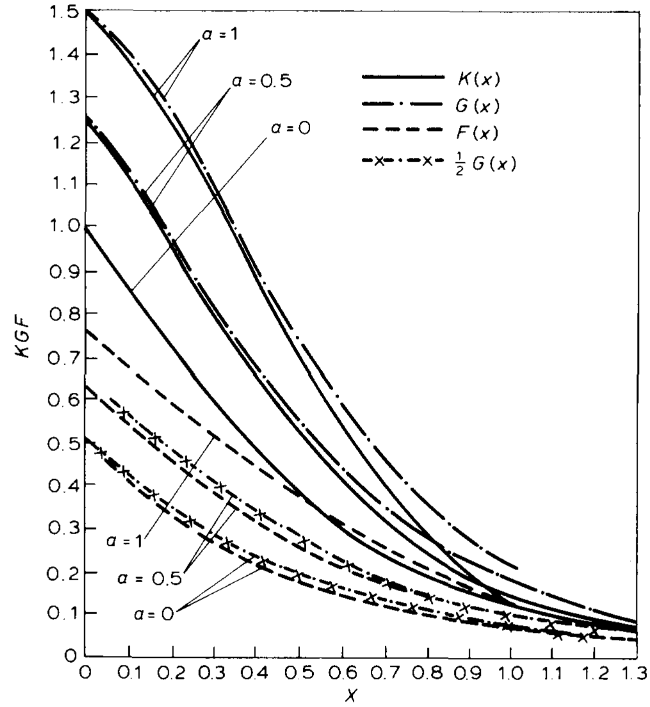

Fig 6 Exact and approximate configuration factors

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